Curvature & Gravitation
Schwarchild & Eddington metrics
Curving flat Electromagnetic Space manifesting Gravitation

Let: $\mathbf{A} = \mathbf{u}_j A^j$, (basis: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ components: A^j),

$$\mathbf{r} \equiv (\mathbf{x}) = C^j_{\mathbf{r}}\mathbf{u}_j \ni d\mathbf{r} = \mathbf{u}_j dx^j$$

and $\exists \mathbf{u}_{j0}$ constants $\ni \mathbf{u}_j = L^m_j \mathbf{u}_{m0}$, $\mathbf{u}_{m0} = \Gamma^h_m \mathbf{u}_h \Rightarrow \delta^h_j = L^m_j \Gamma^h_m$

so:

$$\delta\mathbf{r} = \mathbf{u}_j \delta x^j$$

$$\begin{aligned}
\delta\mathbf{r} &= \delta\left(C^j_{\mathbf{r}}\mathbf{u}_j\right) \\
&= \left(\delta C^j_{\mathbf{r}}\mathbf{u}_j + C^j_{\mathbf{r}}\delta\mathbf{u}_j\right) \\
&= \left(\delta C^j_{\mathbf{r}}\mathbf{u}_j + C^j_{\mathbf{r}}\left(\left(\left(\Gamma^k_m\left(\frac{\partial}{\partial x^h}L^m_j|_{\mathbf{x}=\boldsymbol{\xi}}\right)\delta x^h\right)\mathbf{u}_k\right)\right)\right) \\
&= \left(\delta C^j_{\mathbf{r}} + \left(\left(\left(C^k_{\mathbf{r}}\Gamma^j_m\left(\frac{\partial}{\partial x^h}L^m_k|_{\mathbf{x}=\boldsymbol{\xi}}\right)\delta x^h\right)\right)\mathbf{u}_j\right)\right) \\
&= \left(\left(\frac{\partial C^j_{\mathbf{r}}}{\partial x^h} + \left(\left(C^k_{\mathbf{r}}\Gamma^j_m\left(\frac{\partial}{\partial x^h}L^m_k|_{\mathbf{x}=\boldsymbol{\xi}}\right)\right)\right)\delta x^h \mathbf{u}_j\right)\right)
\end{aligned}$$

for some basis,

$$\mathbf{u}_j \equiv \frac{\partial \mathbf{r}(\boldsymbol{\xi})}{\partial x^i} \text{ at } \boldsymbol{\xi}$$

then

$$\delta\mathbf{r} = \left(\frac{\partial \mathbf{r}(\boldsymbol{\xi})}{\partial x^i}\delta x^i\right) = (\mathbf{u}_j \delta x^i)$$

so, using this basis:

$$\left(\left(\left(\left(C^k_{\mathbf{r}}\Gamma^j_m\left(\frac{\partial}{\partial x^h}L^m_k|_{\mathbf{x}=\boldsymbol{\xi}}\right)\right)\delta x^h \mathbf{u}_j\right)\right) = \mathbf{0}$$

$\therefore$

$$\left(\left(C^k_{\mathbf{r}}\Gamma^j_m\left(\frac{\partial}{\partial x^h}L^m_k|_{\mathbf{x}=\boldsymbol{\xi}}\right)\right)\delta x^h = 0$$

for arbitrary δx^h , so, in the limit, as $\mathbf{x} \to \boldsymbol{\xi}$:

$$\forall j,h \in N \ni 1 \le j,h \le n \quad : \quad C^k_{\mathbf{r}}\left(\Gamma^j_m \frac{\partial}{\partial x^h}L^m_k\right) = 0$$

Accordingly, the **Christoffel symbols of the 2nd kind** are defined:

$$\left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \equiv \Gamma^j_m \frac{\partial}{\partial x^h}L^m_k$$

so:

$$\forall j,h \in N \ni 1 \le j,h \le n \quad : \quad C^k_{\mathbf{r}}\left\{ \begin{matrix} j \\ hk \end{matrix} \right\} = 0$$

constrains the positon vector components.

for any function:

$$
\begin{aligned}
\delta\mathbf{f} &= \delta(f^j\mathbf{u}_j) \\
&= (\delta f^j\mathbf{u}_j + f^j\delta\mathbf{u}_j) \\
&= \delta f^j\mathbf{u}_j + f^j\left(\Gamma^k_m(\frac{\partial}{\partial x^h}L^m_j|_{\mathbf{x}=\boldsymbol{\xi}}\right)\delta x^h\mathbf{u}_k \\
&= \left[\frac{\partial f^j}{\partial x^h} + \left(f^k\Gamma^j_m(\frac{\partial}{\partial x^h}L^m_k|_{\mathbf{x}=\boldsymbol{\xi}}\right)\right]\delta x^h\mathbf{u}_j
\end{aligned}
$$

in the limit, as $\mathbf{x} \to \boldsymbol{\xi}$:

$$
= \left[\frac{\partial f^j}{\partial x^h} + f^k\left(\Gamma^j_m\frac{\partial}{\partial x^h}L^m_k\right)\right]\delta x^h\mathbf{u}_j = \left(\frac{\partial f^j}{\partial x^h} + f^k\left\{\begin{matrix} j \\ hk \end{matrix}\right\}\right)\delta x^h\mathbf{u}_j
$$

Accordingly, the **Covariant Derivative** is defined:

$$
f^j_{;h} \equiv \frac{\partial f^j}{\partial x^h} + f^k\left\{\begin{matrix} j \\ hk \end{matrix}\right\}
$$

so:

$$
\delta\mathbf{f} = f^j_{;h}\delta x^h\mathbf{u}_j
$$

For consistency the following definitions are also made:

the **metric tensor** is defined by:

$$
ds^2 = g_{ij}dx^i dx^j
$$

the **Christoffel symbols of the 1st kind** are defined:

$$
[ik,j] \equiv \tfrac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right)
$$

The following may be found in any differential geometry or general relativity reference, such as [14]:

$$
\left\{\begin{matrix} i \\ jk \end{matrix}\right\} = g^{im}[jk,m]
$$

A geodesic is an extrema of a path between two points in a space.

So:

$$
\begin{aligned}
&0 = \delta\int_{s_1}^{s_2} ds = \delta\int_{s_1}^{s_2}\sqrt{g_{ij}\frac{dx^i}{du}\frac{dx^j}{du}}\,du \\
&\Rightarrow \frac{d^2x^i}{du^2} + \left\{\begin{matrix} i \\ jk \end{matrix}\right\}\frac{dx^j}{du}\frac{dx^k}{du} = 0 \qquad (1)
\end{aligned}
$$

If gravitational motion is considered motion along a geodesic in curved space-time in the absence of any force, then:

$$
-\frac{\partial\varphi_g}{\partial x^h} = F_g = m\frac{d^2x^i}{dt^2} = -m\left\{\begin{matrix} i \\ jk \end{matrix}\right\}\frac{dx^j}{dt}\frac{dx^k}{dt}
$$

So, given the $\left\{ {i \atop jk} \right\}$, or equivalently, the: g_{ij}
gravitational force may be realized, due to the principle of equivalence (of inertial and gravitational mass).

The **Riemann curvature tensor** is defined by:

$$R^i_{jkh} \equiv \frac{\partial}{\partial x^h}\left\{ {i \atop kj} \right\} - \frac{\partial}{\partial x^k}\left\{ {i \atop jh} \right\} + \left\{ {i \atop mh} \right\}\left\{ {m \atop kj} \right\} - \left\{ {i \atop mk} \right\}\left\{ {m \atop hj} \right\}$$

The **Ricci tensor** is the contraction of the Riemann curvature tensor:

$$R_{jh} \equiv R^m_{jmh}$$

The Einstein free-space field equations may be written:

$$R_{jh} = 0$$

or, alternatively:

$$R^j{}_h - \tfrac{1}{2} g^j{}_h R = 0$$

analogously to Laplace's equation:

$$\sum_{j=1}^{3} \frac{\partial^2 \varphi_g}{\partial x^{j2}} = 0$$

$$\Rightarrow 0 = R_{jh} = \frac{\partial}{\partial x^h}\left\{ {n \atop nj} \right\} - \frac{\partial}{\partial x^n}\left\{ {n \atop jh} \right\} + \left\{ {n \atop mh} \right\}\left\{ {m \atop nj} \right\} - \left\{ {n \atop mn} \right\}\left\{ {m \atop hj} \right\} \qquad (2)$$

The Schwarzschild Solution

The usual spherical line element is given by:

$$ds^2 = c^2 dt^2 - \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

Generalizing this form for suitable choice of radial coordinate:

$$ds^2 = A(r)c^2 dt^2 - B(r)dr^2 - \left(r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

where $A(r), B(r)$ are intrinsically positive functions.

Let: $A(r) \equiv e^{\nu(r)}$, $B(r) \equiv e^{\lambda(r)}$
Then, from the extrema:

$$0 = \delta \int \sqrt{e^\nu c^2 dt^2 - e^\lambda dr^2 - \left(r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)}$$

the Euler-Lagrange equations are (for $x^\alpha = x^0 = ct$) :

$$\frac{d}{ds}\left(2e^\nu \frac{dx^0}{ds}\right) = 0$$

$$\Rightarrow \frac{d^2 x^0}{ds^2} + \nu' \dot{r} \frac{dx^0}{ds} = 0$$

Comparing this to (1) :

$$\left\{ \begin{matrix} 0 \\ 10 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ 01 \end{matrix} \right\} = \tfrac{1}{2}\nu'$$

Similarly, for r :

$$\Rightarrow \frac{d}{ds}\left(\frac{dr}{ds}\right) + \tfrac{1}{2}\lambda'\left(\frac{dr}{ds}\right)^2 + \tfrac{1}{2}\nu' e^{\nu-\lambda}\left(\frac{dx^0}{ds}\right)^2 - e^{-\lambda} r\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2 e^{-\lambda} = 0$$

So:

$$\left\{ \begin{matrix} 1 \\ 00 \end{matrix} \right\} = \tfrac{1}{2}\nu' e^{\nu-\lambda} \quad , \quad \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = \tfrac{1}{2}\lambda'$$

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -re^{-\lambda} \quad , \quad \left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} = -r\sin^2\theta e^{-\lambda}$$

Again, for θ :

$$\Rightarrow \frac{d}{ds}\left(\frac{d\theta}{ds}\right) + \frac{2}{r}\frac{d\theta}{ds}\frac{dr}{ds} - \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 = 0$$

So:

$$\left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \frac{1}{r} \quad , \quad \left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} = -\sin\theta\cos\theta$$

And, for φ :

$$\Rightarrow \frac{d}{ds}\left(\frac{d\theta}{ds}\right) + 2\cot\theta\frac{d\varphi}{ds}\frac{d\theta}{ds} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} = 0$$

So:

$$\left\{ \begin{matrix} 3 \\ 23 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 32 \end{matrix} \right\} = \cot\theta \quad , \quad \left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 31 \end{matrix} \right\} = \frac{1}{r}$$

The rest of the Christoffel symbols are zero.
So, from (2) :

$$0 = R_{jh} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^h}\left(\log\sqrt{-g}\right) - \frac{\partial}{\partial x^m}\left\{ \begin{matrix} m \\ jh \end{matrix} \right\} + \left\{ \begin{matrix} n \\ mh \end{matrix} \right\}\left\{ \begin{matrix} m \\ nj \end{matrix} \right\} - \left\{ \begin{matrix} m \\ jh \end{matrix} \right\}\frac{\partial}{\partial x^m}\left(\log\sqrt{-g}\right)$$

Now:

$$g_{ij} = \begin{pmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

So:

$$g = |g_{ij}| = \begin{vmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{vmatrix} = -r^4 e^{\nu(r)+\lambda(r)}\sin^2\theta$$

$$\Rightarrow \log\sqrt{-g} = \tfrac{1}{2}(\nu+\lambda) + 2\log r + \log|\sin\theta|$$

So, from (2) :

$$0 = R_{00} = \frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0}\left(\log\sqrt{-g}\right) - \frac{\partial}{\partial x^n}\left\{ \begin{matrix} n \\ 00 \end{matrix} \right\} +$$

$$+ \left\{ \begin{matrix} n \\ m0 \end{matrix} \right\}\left\{ \begin{matrix} m \\ n0 \end{matrix} \right\} - \left\{ \begin{matrix} n \\ 00 \end{matrix} \right\}\frac{\partial}{\partial x^n}\left(\log\sqrt{-g}\right)$$

$$\Rightarrow 0 = R_{00} = -\frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 00 \end{matrix} \right\} + 2\left\{ \begin{matrix} 1 \\ 00 \end{matrix} \right\}\left\{ \begin{matrix} 0 \\ 01 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 00 \end{matrix} \right\}\frac{\partial}{\partial x^1}\left(\log\sqrt{-g}\right)$$

$$= -\left(\tfrac{1}{2}\nu' e^{\nu-\lambda}\right)' + \left(\tfrac{1}{2}\nu'^2 e^{\nu-\lambda}\right) - \left(\tfrac{1}{2}\nu' e^{\nu-\lambda}\right)\left[\tfrac{1}{2}(\nu+\lambda)' + \frac{2}{r}\right]$$

$$= -\tfrac{1}{2}e^{\nu-\lambda}\left(\nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\nu'\lambda' + \frac{2\nu'}{r}\right)$$

$$\Rightarrow 0 = \nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\nu'\lambda' + \frac{2\nu'}{r}$$

Similarly:

$$0 = R_{11} = \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}\left(\log\sqrt{-g}\right) - \frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} + \left\{ \begin{matrix} 0 \\ 10 \end{matrix} \right\}\left\{ \begin{matrix} 0 \\ 10 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} +$$

$$+ \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\}\left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 31 \end{matrix} \right\}\left\{ \begin{matrix} 3 \\ 31 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\}\frac{\partial}{\partial x^1}\left(\log\sqrt{-g}\right)$$

$$= \tfrac{1}{2}(\nu+\lambda)'' - \frac{2}{r^2} - \tfrac{1}{2}\lambda'' + \tfrac{1}{4}\nu'^2 + \frac{2}{r^2} - \tfrac{1}{2}\lambda'\left[\tfrac{1}{2}(\nu+\lambda)' - \frac{2}{r}\right]$$

$$= -\tfrac{1}{2}\left(\nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\nu'\lambda' + \frac{2\lambda'}{r}\right)$$

$$\Rightarrow 0 = \nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\nu'\lambda' + \frac{2\lambda'}{r}$$

Subtracting:

$$(\nu+\lambda)' = 0 \Rightarrow \nu + \lambda = k$$

and k may be chosen to be 0, so:

$$\lambda = -\nu$$

Yielding:

$$0 = \lambda'' - \lambda'^2 + \frac{2\lambda'}{r}$$

$$\Rightarrow 0 = \left(re^{-\lambda}\right)''$$

Continuing, from (2) :

$$0 = R_{22} = \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}\left(\log\sqrt{-g}\right) - \frac{\partial}{\partial x^n}\left\{ \begin{matrix} n \\ 22 \end{matrix} \right\} +$$

$$+ \left\{ \begin{matrix} n \\ m2 \end{matrix} \right\}\left\{ \begin{matrix} m \\ n2 \end{matrix} \right\} - \left\{ \begin{matrix} n \\ 22 \end{matrix} \right\}\frac{\partial}{\partial x^n}\left(\log\sqrt{-g}\right)$$

$$\Rightarrow 0 = R_{22} = \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}\left(\log\sqrt{-g}\right) - \frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\}\left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} + \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} +$$

$$+ \left\{ \begin{matrix} 3 \\ 23 \end{matrix} \right\}\left\{ \begin{matrix} 3 \\ 23 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\}\frac{\partial}{\partial x^1}\left(\log\sqrt{-g}\right)$$

$$= \frac{\partial^2}{\partial\theta^2}(\log|\sin\theta|) + \left(re^{-\lambda}\right)' + 2\left(-e^{-\lambda}\right) + \cot^2\theta + re^{-\lambda}\left[\tfrac{1}{2}(\nu+\lambda)' + \frac{2}{r}\right]re^{-\lambda}$$

$$\Rightarrow 1 = \left(re^{-\lambda}\right)'$$

$$\Rightarrow re^{-\lambda} = r - 2m$$

$$\Rightarrow e^{\nu} = 1 - \frac{2m}{r} \quad , \quad e^{\lambda} = \frac{1}{1 - \frac{2m}{r}}$$

And:

$$0 = R_{33} = \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3}\left(\log\sqrt{-g}\right) - \frac{\partial}{\partial x^n}\left\{ \begin{matrix} n \\ 33 \end{matrix} \right\} +$$

$$+ \left\{ \begin{matrix} n \\ m3 \end{matrix} \right\}\left\{ \begin{matrix} m \\ n3 \end{matrix} \right\} - \left\{ \begin{matrix} n \\ 33 \end{matrix} \right\}\frac{\partial}{\partial x^n}\left(\log\sqrt{-g}\right)$$

$$\Rightarrow 0 = R_{22} = -\frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} - \frac{\partial}{\partial x^2}\left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} + 2\left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} + 2\left\{ \begin{matrix} 3 \\ 32 \end{matrix} \right\}\left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} +$$
$$-\left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\}\frac{\partial}{\partial x^1}\left(\log\sqrt{-g}\right) - \left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\}\frac{\partial}{\partial x^2}\left(\log\sqrt{-g}\right)$$
$$\Rightarrow 0 = \left(re^{-\lambda}\sin^2\theta\right)' + \frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 2\left(-e^{-\lambda}\sin^2\theta - \cot\theta\sin\theta\cos\theta\right)$$
$$+ re^{-\lambda}\sin^2\theta\left(\frac{2}{r}\right) + \cos^2\theta$$

since:

$$\log\sqrt{-g} = 2\log r + \log|\sin\theta|$$
$$\Rightarrow 0 = \sin^2\theta\left[(re^{-\lambda})' - 1\right]$$

which is identically true.

By appealing to the Newtonian theory, the constant m may be determined:

$$\varphi = -\frac{GM}{r} \Rightarrow g_{00} \simeq 1 + \frac{2\varphi}{c^2} \Rightarrow m = \frac{1}{c^2}GM$$

Yielding the Schwarzschild line element:

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2dt^2 - (1/(1-2m/r))dr^2 - \left(r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right)$$

The transformation:

$$\overline{t} = t + \frac{2m}{c}\log\left|\frac{r}{2m} - 1\right|$$

yields the Eddington form of the Schwarzschild line element:

$$ds^2 = c^2d\overline{t}^2 - \left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right) - \frac{2m}{r}(cdt + dr)^2$$

In Cartesian coordinates it is:

$$ds^2 = c^2d\overline{t}^2 - dx^2 - dy^2 - dz^2 - \frac{2m}{r}(cdt + dr)^2,$$
$$r = \sqrt{x^2 + y^2 + z^2}$$

The gravitational field is manifested through curvature of the flat electromagnetic space, as follows.

Given: $\overline{x} = x \;,\; \overline{y} = y \;,\; \overline{z} = z \;,\; \overline{t} = t + \frac{2m}{c}\log\left|\frac{r}{2m} - 1\right| \;,\; \left(r = \sqrt{x^2 + y^2 + z^2}\right)$

$$\frac{\partial\mathbf{r}}{\partial\overline{x}} = \mathbf{v}^{4;1} \;,\; \mathbf{u}^{4;1} = \frac{\partial\mathbf{r}}{\partial\overline{x}}\frac{\partial\overline{x}}{\partial x} + \frac{\partial\mathbf{r}}{\partial\overline{y}}\frac{\partial\overline{y}}{\partial x} + \frac{\partial\mathbf{r}}{\partial\overline{z}}\frac{\partial\overline{z}}{\partial x} + \frac{\partial\mathbf{r}}{\partial\overline{t}}\frac{\partial\overline{t}}{\partial x} = \mathbf{v}^{4;1} + \mathbf{v}^{4;0}\frac{\partial\overline{t}}{\partial x}$$
$$= \mathbf{v}^{4;1} + \mathbf{v}^{4;0}\frac{1}{c}\frac{1}{\left|\frac{r}{2m} - 1\right|}\frac{x}{r^2}$$

$$\frac{\partial\mathbf{r}}{\partial\overline{y}} = \mathbf{v}^{4;2} \;,\; \mathbf{u}^{4;2} = \frac{\partial\mathbf{r}}{\partial\overline{x}}\frac{\partial\overline{x}}{\partial y} + \frac{\partial\mathbf{r}}{\partial\overline{y}}\frac{\partial\overline{y}}{\partial y} + \frac{\partial\mathbf{r}}{\partial\overline{z}}\frac{\partial\overline{z}}{\partial y} + \frac{\partial\mathbf{r}}{\partial\overline{t}}\frac{\partial\overline{t}}{\partial y} = \mathbf{v}^{4;2} + \mathbf{v}^{4;0}\frac{\partial\overline{t}}{\partial y}$$
$$= \mathbf{v}^{4;2} + \mathbf{v}^{4;0}\frac{1}{c}\frac{1}{\left|\frac{r}{2m} - 1\right|}\frac{y}{r^2}$$

$$\frac{\partial\mathbf{r}}{\partial\overline{z}} = \mathbf{v}^{4;3} \;,\; \mathbf{u}^{4;3} = \frac{\partial\mathbf{r}}{\partial\overline{x}}\frac{\partial\overline{x}}{\partial z} + \frac{\partial\mathbf{r}}{\partial\overline{y}}\frac{\partial\overline{y}}{\partial z} + \frac{\partial\mathbf{r}}{\partial\overline{z}}\frac{\partial\overline{z}}{\partial z} + \frac{\partial\mathbf{r}}{\partial\overline{t}}\frac{\partial\overline{t}}{\partial z} = \mathbf{v}^{4;3} + \mathbf{v}^{4;0}\frac{\partial\overline{t}}{\partial z}$$

$$= \mathbf{v}^{4;3} + \mathbf{v}^{4;0}\frac{1}{c}\frac{1}{\left|\frac{r}{2m}-1\right|}\frac{z}{r^2}$$

$$\frac{\partial \mathbf{r}}{\partial \overline{t}} = \mathbf{v}^{4;0} \quad , \quad \mathbf{u}^{4;0} = \frac{\partial \mathbf{r}}{\partial \overline{x}}\frac{\partial \overline{x}}{\partial t} + \frac{\partial \mathbf{r}}{\partial \overline{y}}\frac{\partial \overline{y}}{\partial t} + \frac{\partial \mathbf{r}}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial t} + \frac{\partial \mathbf{r}}{\partial \overline{t}}\frac{\partial \overline{t}}{\partial t} = \mathbf{v}^{4;0}$$

$\therefore$

$$\mathbf{v}^{4;1} = \mathbf{u}^{4;1} - \mathbf{u}^{4;0}\frac{1}{c}\frac{1}{\left|\frac{r}{2m}-1\right|}\frac{x}{r^2}$$

$$\mathbf{v}^{4;2} = \mathbf{u}^{4;1} - \mathbf{u}^{4;0}\frac{1}{c}\frac{1}{\left|\frac{r}{2m}-1\right|}\frac{y}{r^2}$$

$$\mathbf{v}^{4;3} = \mathbf{u}^{4;1} - \mathbf{u}^{4;0}\frac{1}{c}\frac{1}{\left|\frac{r}{2m}-1\right|}\frac{z}{r^2}$$

$$\mathbf{v}^{4;0} = \mathbf{u}^{4;0}$$

So:

$$\mathbf{u}^{4;k} = \mathbf{v}^{4;k} + \mathbf{v}^{4;0}\frac{1}{c}\frac{(1-\delta^k_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^k}{r^2}$$

$$\mathbf{v}^{4;k} = \mathbf{u}^{4;k} - \mathbf{u}^{4;0}\frac{1}{c}\frac{(1-\delta^k_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^k}{r^2}$$

$$\Rightarrow \overline{\beta}^{\,h}_{mn}\mathbf{v}^{4;h} = \mathbf{v}^{4;m}\circ\mathbf{v}^{4;n} = \left(\mathbf{u}^{4;m} - \mathbf{u}^{4;0}\frac{1}{c}\frac{(1-\delta^m_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^m}{r^2}\right)\circ\left(\mathbf{u}^{4;n} - \mathbf{u}^{4;0}\frac{1}{c}\frac{(1-\delta^n_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^n}{r^2}\right)$$

$$= \mathbf{u}^{4;m}\circ\mathbf{u}^{4;n} - \mathbf{u}^{4;m}\circ\mathbf{u}^{4;0}\frac{1}{c}\frac{(1-\delta^n_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^n}{r^2} - \mathbf{u}^{4;0}\circ\mathbf{u}^{4;n}\frac{1}{c}\frac{(1-\delta^m_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^m}{r^2} +$$

$$+ \mathbf{u}^{4;0}\circ\mathbf{u}^{4;0}\frac{1}{c^2}(1-\delta^m_0)(1-\delta^n_0)\frac{1}{\left|\frac{r}{2m}-1\right|^2}\frac{x^m}{r^2}\frac{x^n}{r^2}$$

$$= \beta^k_{mn}\mathbf{u}^{4;k} - \beta^k_{m0}\mathbf{u}^{4;k}\frac{1}{c}\frac{(1-\delta^n_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^n}{r^2} - \beta^k_{0n}\mathbf{u}^{4;k}\frac{1}{c}\frac{(1-\delta^m_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^m}{r^2} +$$

$$+ \beta^k_{00}\mathbf{u}^{4;k}\frac{1}{c^2}(1-\delta^m_0)(1-\delta^n_0)\frac{1}{\left|\frac{r}{2m}-1\right|^2}\frac{x^m}{r^2}\frac{x^n}{r^2}$$

$$= \mathbf{u}^{4;k}\Big[\beta^k_{mn} - \beta^k_{m0}\frac{1}{c}\frac{(1-\delta^n_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^n}{r^2} - \beta^k_{0n}\frac{1}{c}\frac{(1-\delta^m_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^m}{r^2} +$$

$$+ \beta^k_{00}\frac{1}{c^2}(1-\delta^m_0)(1-\delta^n_0)\frac{1}{\left|\frac{r}{2m}-1\right|^2}\frac{x^m}{r^2}\frac{x^n}{r^2}\Big]$$

$$= \left(\mathbf{v}^{4;k} + \mathbf{v}^{4;0}\frac{1}{c}\frac{(1-\delta^k_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^k}{r^2}\right)\Big[\beta^k_{mn} - \beta^k_{m0}\frac{1}{c}\frac{(1-\delta^n_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^n}{r^2} +$$

$$- \beta^k_{0n}\frac{1}{c}\frac{(1-\delta^m_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^m}{r^2} +$$

$$+\beta^k_{00}\frac{1}{c^2}(1-\delta^m_0)(1-\delta^n_0)\frac{1}{\left|\frac{r}{2m}-1\right|^2}\frac{x^m}{r^2}\frac{x^n}{r^2}\Big]$$

$$= \mathbf{v}^{4;h}\left(\delta^k_h+\delta^0_h\frac{1}{c}\frac{(1-\delta^k_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^k}{r^2}\right)\Big[\beta^k_{mn}-\beta^k_{m0}\frac{1}{c}\frac{(1-\delta^n_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^n}{r^2}+$$
$$-\beta^k_{0n}\frac{1}{c}\frac{(1-\delta^m_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^m}{r^2}+$$
$$+\beta^k_{00}\frac{1}{c^2}(1-\delta^m_0)(1-\delta^n_0)\frac{1}{\left|\frac{r}{2m}-1\right|^2}\frac{x^m}{r^2}\frac{x^n}{r^2}\Big]$$

So:

$$\overline{\beta}^{\,h}_{mn} = \left(\delta^k_h+\delta^0_h\frac{1}{c}\frac{(1-\delta^k_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^k}{r^2}\right)\Big[\beta^k_{mn}-\beta^k_{m0}\frac{1}{c}\frac{(1-\delta^n_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^n}{r^2}+$$
$$-\beta^k_{0n}\frac{1}{c}\frac{(1-\delta^m_0)}{\left|\frac{r}{2m}-1\right|}\frac{x^m}{r^2}+$$
$$+\beta^k_{00}\frac{1}{c^2}(1-\delta^m_0)(1-\delta^n_0)\frac{1}{\left|\frac{r}{2m}-1\right|^2}\frac{x^m}{r^2}\frac{x^n}{r^2}\Big]$$

Thus, using these in the construction of the weighted matrix product for:
$\mathbf{v}^{4;m}\circ\mathbf{v}^{4;n} = \overline{\beta}^{\,h}_{mn}\mathbf{v}^{4;h}$ (given: $\mathbf{u}^{4;m}\circ\mathbf{u}^{4;n} = \beta^h_{mn}\mathbf{u}^{4;h}$)
the gravitational field is manifested as a special case of the electromagnetic field, where the three-vector potential (f^1,f^2,f^3)
vanishes everywhere, leaving: $g_{00}\equiv\varphi_g\equiv(f^0,0,0,0)$ as a scalar field (transformed away as curvature of the ”space-time” -which is
why they satisfy the Einstein field equations: $0 = R_{ij} = \Box\varphi_g$), with coupling constant (G) constrained by the principle of equivalence.

References

[14] Adler, R. & Bazin, M. & Schiffer, M.; ”*Introduction to General Relativity, 2nd Ed.*”, McGraw-Hill Book Company; New York; 1975.