

**Quaternions
from
Bare Matrices
using the
Weighted Matrix Product**

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A Quaternion space may be generated using the vector basis:

$$\hat{\mathbf{i}} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{\mathbf{j}} \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{k}} \equiv \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\mathbf{t}} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Under ordinary Matrix Multiplication,
the products Cayley table becomes:

-	$\hat{\mathbf{t}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
$\hat{\mathbf{i}}$	$\hat{\mathbf{i}}$	$-\hat{\mathbf{t}}$	$\hat{\mathbf{k}}$	$-\hat{\mathbf{j}}$
$\hat{\mathbf{j}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{k}}$	$-\hat{\mathbf{t}}$	$\hat{\mathbf{i}}$
$\hat{\mathbf{k}}$	$\hat{\mathbf{k}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{i}}$	$-\hat{\mathbf{t}}$

and, so:

$$\begin{aligned} \mathbf{A} \circ \mathbf{B} &= \hat{\mathbf{t}}(A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3) + \\ &+ \hat{\mathbf{i}}(A^0B^1 + A^1B^0 + A^2B^3 - A^3B^2) + \\ &+ \hat{\mathbf{j}}(A^0B^2 - A^1B^3 + A^2B^0 + A^3B^1) + \\ &+ \hat{\mathbf{k}}(A^0B^3 + A^1B^2 - A^2B^1 + A^3B^0) \end{aligned}$$

∴

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &\equiv \frac{1}{2}(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) \\ &= \hat{\mathbf{i}}(A^2B^3 - A^3B^2) + \\ &+ \hat{\mathbf{j}}(-A^1B^3 + A^3B^1) + \\ &+ \hat{\mathbf{k}}(A^1B^2 - A^2B^1) \\ \mathbf{A} \odot \mathbf{B} &\equiv \frac{1}{2}(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A}) \end{aligned}$$

$$\begin{aligned}
&= \hat{\mathbf{t}}(A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3) + \\
&+ \hat{\mathbf{i}}(A^0B^1 + A^1B^0) + \\
&+ \hat{\mathbf{j}}(A^0B^2 + A^2B^0) + \\
&+ \hat{\mathbf{k}}(A^1B^2 - A^2B^1)
\end{aligned}$$

This Cayley product, can also be obtained using Bare Matrices with appropriate weight matrices and weighted matrix multiplication.

Using the bare matrices:

$$\begin{aligned}
\mathbf{u}^{4;0} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \mathbf{u}^{4;1} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\mathbf{u}^{4;2} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \mathbf{u}^{4;3} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
\delta_j^i &\equiv \begin{cases} =1, & \text{iff: } i=j \\ =0, & \text{iff: } i \neq j \end{cases} \quad (i, j \in \mathbf{R}), \quad \text{Kronecker Delta}
\end{aligned}$$

$$\sum_{i \geq X}^Y A(i) \equiv \begin{cases} = \sum_{i=X}^Y A(i), & \forall Y \geq X \\ = 0, & \forall Y < X \end{cases}$$

$$T_0(i) \equiv \frac{1}{2}(i - 1 + \delta_{(-1)^i}^1), \quad (i \in \mathbf{N})$$

$$S_n(i) \equiv i + 2^n \cdot (-1)^{T_{n-1}(i)} \cdot (1 - \delta_0^i), \quad (i, n \in \mathbf{N})$$

$$I(h) \equiv \sum_{r \geq 0}^{h-1} \delta_{T_r(h)}^1 \cdot (r+1), \quad (h \in \mathbf{N} \geq 1)$$

$$J_h \equiv \begin{cases} = \{\}, & (h \in \{0, 1\}) \\ = \{I(h)\}, & (h \in \{2, 3, 4, 5\}) \\ = \{I(h), y \mid y \in I(h - 2^{I(h)})\}, & (h \in \mathbf{N} \geq 6) \end{cases}$$

$$L_h \equiv \begin{cases} = \{\}, & (h = 0) \\ = J_{h-2^{I(h)}}, & (h \in \mathbf{N} \geq 1) \end{cases}$$

$$m_0(k, h, i) \equiv S_k(i) + \sum_{r \in L_{2^k+h}} 2^r \cdot (-1)^{T_{r-1}(i)} + (1 - \delta_{(-1)^h}^1) \cdot (-1)^i,$$

$$(\forall i, h, k \in \mathbf{N} \ni I(h + 2^k \cdot [1 - \delta_k^0]) =$$

$$w(k, i) \equiv \begin{cases} 0, & (\text{iff} : i = 0) \\ w(k, i-1) + (1 - \delta_2^k) \cdot \left(1 - \delta_{(-1)^i}^1\right) \cdot 2^{k-1} \cdot (-1)^{\binom{i-2^0}{2^1}} + \\ \quad + \sum_{j \geq 0}^{k-4} \delta_{(-1)^{2^j}}^1 \cdot \left(1 - \delta_{(-1)^{2^{j+1}}}^1\right) \cdot 2^{k-1-(1+j)} \cdot (-1)^{\binom{i-2^{0+(1+j)}}{2^{1+(1+j)}}} + \\ \quad + \sum_{j \geq 1}^3 \delta_{j \cdot 2^{k-2}}^i \cdot (1 - \delta_1^k), & (\text{iff} : i \neq 0) \end{cases}$$

$$\mathbf{u}^{2^k; w(k, h)} \equiv \left(\mathbf{u}_{ij}^{2^k; w(k, h)}\right) \ni \mathbf{u}_{ij}^{2^k; w(k, h)} = \delta_{m(k, I(h), h-2^{I(h)}, (1-\delta_0^{I(h)}), i)}^j$$

$$(\forall h, k \in \mathbf{N} \ni 0 \leq j \leq k, 0 \leq h \leq 2)$$

i	0	1	2	3
$w(2, i)$	0	1	2	3

$\left(\mathbf{u}_{ij}^{4;0}\right) = \delta_{m_0(0,0,i)}^j = \delta_{ij}$	$\left(\mathbf{u}_{ij}^{4;1}\right) = \delta_{m_0(0,1,i)}^j$
$\left(\mathbf{u}_{ij}^{4;2}\right) = \delta_{m_0(1,0,i)}^j$	$\left(\mathbf{u}_{ij}^{4;3}\right) = \delta_{m_0(1,1,i)}^j$

The weight matrices are fully determined by:

$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;0}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;1}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;0}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;2}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;2}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;3}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;2}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;0}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;0}$

equivalent to the Cayley table:

–	$\hat{\mathbf{t}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
$\hat{\mathbf{i}}$	$\hat{\mathbf{i}}$	$-\hat{\mathbf{t}}$	$\hat{\mathbf{k}}$	$-\hat{\mathbf{j}}$
$\hat{\mathbf{j}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{k}}$	$-\hat{\mathbf{t}}$	$\hat{\mathbf{i}}$
$\hat{\mathbf{k}}$	$\hat{\mathbf{k}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{i}}$	$-\hat{\mathbf{t}}$

–	$\mathbf{u}^{4;0}$	$\mathbf{u}^{4;1}$	$\mathbf{u}^{4;2}$	$\mathbf{u}^{4;3}$
$\mathbf{u}^{4;0}$	$\mathbf{u}^{4;0}$	$\mathbf{u}^{4;1}$	$\mathbf{u}^{4;2}$	$\mathbf{u}^{4;3}$
$\mathbf{u}^{4;1}$	$\mathbf{u}^{4;1}$	$-\mathbf{u}^{4;0}$	$\mathbf{u}^{4;3}$	$-\mathbf{u}^{4;2}$
$\mathbf{u}^{4;2}$	$\mathbf{u}^{4;2}$	$-\mathbf{u}^{4;3}$	$-\mathbf{u}^{4;0}$	$\mathbf{u}^{4;1}$
$\mathbf{u}^{4;3}$	$\mathbf{u}^{4;3}$	$\mathbf{u}^{4;2}$	$-\mathbf{u}^{4;1}$	$-\mathbf{u}^{4;0}$

$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;0}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,0,i)}^h \delta_{m_0(0,0,h)}^j$ $1 = \Phi_{iim_0(0,0,i)} = \Phi_{iii}$	$1 = \Phi_{000} = \Phi_{111} = \Phi_{222} = \Phi_{333}$ $= \varphi_{00}^{4;0} = \varphi_{11}^{4;1} = \varphi_{22}^{4;2} = \varphi_{33}^{4;3}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,0,i)}^h \delta_{m_0(0,1,h)}^j$ $1 = \Phi_{iim_0(0,1,i)}$	$1 = \Phi_{001} = \Phi_{110} = \Phi_{223} = \Phi_{332}$ $= \varphi_{01}^{4;0} = \varphi_{10}^{4;1} = \varphi_{23}^{4;2} = \varphi_{32}^{4;3}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,0,i)}^h \delta_{m_0(1,0,h)}^j$ $1 = \Phi_{iim_0(1,0,i)}$	$1 = \Phi_{002} = \Phi_{113} = \Phi_{220} = \Phi_{331}$ $= \varphi_{02}^{4;0} = \varphi_{13}^{4;1} = \varphi_{20}^{4;2} = \varphi_{31}^{4;3}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,0,i)}^h \delta_{m_0(1,1,h)}^j$ $1 = \Phi_{iim_0(1,1,i)}$	$1 = \Phi_{003} = \Phi_{112} = \Phi_{221} = \Phi_{330}$ $= \varphi_{03}^{4;0} = \varphi_{12}^{4;1} = \varphi_{21}^{4;2} = \varphi_{30}^{4;3}$

$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;1}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,1,i)}^h \delta_{m_0(0,0,h)}^j$ $1 = \Phi_{m_0(0,1,i)im_0(0,0,m_0(0,1,i))}$ $1 = \Phi_{m_0(0,1,i)im_0(0,1,i)}$	$1 = \Phi_{101} = \Phi_{010} = \Phi_{323} = \Phi_{232}$ $= \varphi_{01}^{4;1} = \varphi_{10}^{4;0} = \varphi_{23}^{4;3} = \varphi_{32}^{4;2}$
$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;0}$ $-1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,1,i)}^h \delta_{m_0(0,1,h)}^j$ $-1 = \Phi_{m_0(0,1,i)im_0(0,1,m_0(0,1,i))}$ $= \Phi_{m_0(0,1,i)im_0(0,0,i)} = \Phi_{m_0(0,1,i)ii}$	$-1 = \Phi_{100} = \Phi_{011} = \Phi_{322} = \Phi_{233}$ $= \varphi_{00}^{4;1} = \varphi_{11}^{4;0} = \varphi_{22}^{4;3} = \varphi_{33}^{4;2}$
$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,1,i)}^h \delta_{m_0(1,0,h)}^j$ $1 = \Phi_{m_0(0,1,i)im_0(1,0,m_0(0,1,i))}$ $= \Phi_{m_0(0,1,i)im_0(1,1,i)}$	$1 = \Phi_{103} = \Phi_{012} = \Phi_{321} = \Phi_{230}$ $= \varphi_{03}^{4;1} = \varphi_{12}^{4;0} = \varphi_{21}^{4;3} = \varphi_{30}^{4;2}$
$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;2}$ $-1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(0,1,i)}^h \delta_{m_0(1,1,h)}^j$ $-1 = \Phi_{m_0(0,1,i)im_0(1,0,i)}$	$-1 = \Phi_{102} = \Phi_{013} = \Phi_{320} = \Phi_{231}$ $= \varphi_{02}^{4;1} = \varphi_{13}^{4;0} = \varphi_{20}^{4;3} = \varphi_{31}^{4;2}$

$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;2}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,0,i)}^h \delta_{m_0(0,0,h)}^j$ $1 = \Phi_{m_0(1,0,i)im_0(0,0,m_0(1,0,i))}$ $1 = \Phi_{m_0(1,0,i)im_0(1,0,i)}$	$1 = \Phi_{202} = \Phi_{313} = \Phi_{020} = \Phi_{131}$ $= \varphi_{02}^{4;2} = \varphi_{13}^{4;3} = \varphi_{20}^{4;0} = \varphi_{31}^{4;1}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$ $-1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,0,i)}^h \delta_{m_0(0,1,h)}^j$ $-1 = \Phi_{m_0(1,0,i)im_0(0,1,m_0(1,0,i))}$ $= \Phi_{m_0(1,0,i)im_0(1,1,i)}$	$-1 = \Phi_{203} = \Phi_{312} = \Phi_{021} = \Phi_{130}$ $= \varphi_{03}^{4;2} = \varphi_{12}^{4;3} = \varphi_{21}^{4;0} = \varphi_{30}^{4;1}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;0}$ $-1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,0,i)}^h \delta_{m_0(1,0,h)}^j$ $-1 = \Phi_{m_0(1,0,i)im_0(1,0,m_0(1,0,i))}$ $= \Phi_{m_0(1,0,i)im_0(0,0,i)} = \Phi_{m_0(1,0,i)ii}$	$-1 = \Phi_{200} = \Phi_{311} = \Phi_{022} = \Phi_{133}$ $= \varphi_{00}^{4;2} = \varphi_{11}^{4;3} = \varphi_{22}^{4;0} = \varphi_{33}^{4;1}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,0,i)}^h \delta_{m_0(1,1,h)}^j$ $1 = \Phi_{m_0(1,0,i)im_0(1,1,m_0(1,0,i))}$ $1 = \Phi_{m_0(1,0,i)im_0(0,1,i)}$	$1 = \Phi_{201} = \Phi_{310} = \Phi_{023} = \Phi_{132}$ $= \varphi_{01}^{4;2} = \varphi_{10}^{4;3} = \varphi_{23}^{4;0} = \varphi_{32}^{4;1}$

$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;3}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,1,i)}^h \delta_{m_0(0,0,h)}^j$ $1 = \Phi_{m_0(1,1,i)im_0(0,0,m_0(1,1,i))}$ $1 = \Phi_{m_0(1,1,i)im_0(1,1,i)}$	$1 = \Phi_{303} = \Phi_{212} = \Phi_{121} = \Phi_{030}$ $= \varphi_{03}^{4;3} = \varphi_{12}^{4;2} = \varphi_{21}^{4;1} = \varphi_{30}^{4;0}$
$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;2}$ $1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,1,i)}^h \delta_{m_0(0,1,h)}^j$ $1 = \Phi_{m_0(1,1,i)im_0(0,1,m_0(1,1,i))}$ $= \Phi_{m_0(1,1,i)im_0(1,0,i)}$	$1 = \Phi_{302} = \Phi_{213} = \Phi_{120} = \Phi_{031}$ $= \varphi_{02}^{4;3} = \varphi_{13}^{4;2} = \varphi_{20}^{4;1} = \varphi_{31}^{4;0}$
$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$ $-1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,1,i)}^h \delta_{m_0(1,0,h)}^j$ $-1 = \Phi_{m_0(1,1,i)im_0(1,0,m_0(1,1,i))}$ $= \Phi_{m_0(1,1,i)im_0(0,1,i)}$	$-1 = \Phi_{301} = \Phi_{210} = \Phi_{123} = \Phi_{032}$ $= \varphi_{01}^{4;3} = \varphi_{10}^{4;2} = \varphi_{23}^{4;1} = \varphi_{32}^{4;0}$
$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;0}$ $-1 = \sum_{h=0}^3 \Phi_{hij} \delta_{m_0(1,1,i)}^h \delta_{m_0(1,1,h)}^j$ $-1 = \Phi_{m_0(1,1,i)im_0(1,1,m_0(1,1,i))}$ $-1 = \Phi_{m_0(1,1,i)im_0(0,0,i)} = \Phi_{m_0(1,1,i)ii}$	$-1 = \Phi_{300} = \Phi_{211} = \Phi_{122} = \Phi_{033}$ $= \varphi_{00}^{4;3} = \varphi_{11}^{4;2} = \varphi_{22}^{4;1} = \varphi_{33}^{4;0}$

So:

$$\varphi^{4;0} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \varphi^{4;1} = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix},$$

$$\varphi^{4;2} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad \varphi^{4;3} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

thus, it is shown that the bare matrix basis yields a quaternion space under weighted matrix multiplication, with the above

weighted matrices.