The Weighted Matrix Product

Weighted Matrix Multiplication

with applications

This article presents the
Weighted Matrix Product / Weighted Matrix Multiplication
a generalization of ordinary matrix multiplication
with applications, including the four dimensional dot and cross product for Lorentz-Minkowski space

$\mathbf{A} \equiv (A_{ij})$, $\mathbf{B} \equiv (B_{jh})$

$\mathbf{A} \circ \mathbf{B} \equiv (\sum_{\alpha=1}^{n} A_{i\alpha} B_{\alpha j})$

is defined only if:
the number of columns of **A** equals the number of rows of **B**
and then these matrices are termed: **conformable**.

The resultant matrix has the number of rows of the left operand (**A**), and the number of columns of the right operand (**B**).

The Weighted Matrix Product, Weighted Matrix Multiplication
is a generalization of ordinary matrix multiplication, as follows:

Given a set of Weight Matrices, $\mathbf{\Phi}_\alpha \equiv (\Phi_{\alpha ij})$
the Weighted Matrix Product of the matrix pair $(\mathbf{A}, \mathbf{B})$
is given by:

$\mathbf{A} \equiv (A_{ij})$, $\mathbf{B} \equiv (B_{jh})$

$\mathbf{A} \circ \mathbf{B} \equiv (\sum_{\alpha=1}^{c(A)} \Phi_{\alpha ij} A_{i\alpha} B_{\alpha j})$

where:
$c(A)$: the number of columns of **A**

The number of Weight Matrices is:
the number of columns of the left operand =
the number of rows of the right operand
The number of rows of the Weight Matrices is:
the number of rows of the left operand.
The number of columns of the Weight Matrices is:
the number of columns of the right operand.
(conformability is unchanged).

NOTE:
Ordinary Matrix Multiplication is the special case of Weighted Matrix Multiplication, where all the weight matrix entries are $1s$.
Ordinary Matrix Multiplication is Weighted Matrix Multiplication

in a default "sea of 1s", the weight matrices formed out of the "sea" as necessary.

NOTE:

The Weighted Matrix Product is not generally associative:

for:

$$\mathbf{A} \equiv (A_{ij}) \quad , \quad \mathbf{B} \equiv (B_{jh}) \quad , \quad \mathbf{C} \equiv (C_{jh})$$

$$(\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} \equiv \left(\sum_{\alpha=1}^{c(A)} \Phi_{\alpha ij} A_{i\alpha} B_{\alpha j} \right) \circ \mathbf{C} = \sum_{\alpha=1}^{c(A)} \Phi_{\alpha ij} A_{i\alpha} \left(\sum_{\beta=1}^{r(C)} \Phi_{\beta\alpha j} B_{\alpha\beta} C_{\beta j} \right)$$

$$= \sum_{\alpha=1}^{c(A)} \sum_{\beta=1}^{r(C)} \Phi_{\alpha ij} A_{i\alpha} B_{\alpha\beta} C_{\beta j} (\Phi_{\beta\alpha j})$$

$$\mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}) \equiv \mathbf{A} \circ \left(\sum_{\alpha=1}^{c(B)} \Phi_{\alpha ij} B_{i\alpha} C_{\alpha j} \right) = \sum_{\beta=1}^{r(C)} \Phi_{\alpha ij} \left(\sum_{\alpha=1}^{c(A)} \Phi_{\alpha i\beta} A_{i\alpha} B_{\alpha\beta} \right) C_{\beta j}$$

$$= \sum_{\alpha=1}^{c(A)} \sum_{\beta=1}^{r(C)} \Phi_{\alpha ij} A_{i\alpha} B_{\alpha\beta} C_{\beta j} (\Phi_{\alpha i\beta})$$

where:

$c(A)$: the number of columns of $\mathbf{A}$

$r(C)$: the number of rows of $\mathbf{C}$

Weighted matrix multiplication may be expressed in terms of ordinary matrix multiplication, using matrices constructed from the constituent parts, as follows:

for mxp matrix: $\mathbf{A} \equiv (a_{mp})$, and pxn matrix: $\mathbf{B} \equiv (b_{pn})$:

define:

$$\mathbf{A}_i \equiv \begin{pmatrix} a_{1i} & \cdots & \cdots & 0 \\ 0 & a_{2i} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{mi} \end{pmatrix} \quad , \text{ are } mxm \text{ matrices}$$

$$\mathbf{B}_i \equiv \begin{pmatrix} b_{i1} & \cdots & \cdots & 0 \\ 0 & b_{i2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & b_{im} \end{pmatrix} \quad , \text{ are } nxn \text{ matrices}$$

and:

$\mathbf{\Phi}_i$ are mxn matrices

then:

$$\mathbf{A} \circ \mathbf{B} \equiv \left(\sum_{\alpha=1}^{c(A)} \Phi_{\alpha ij} A_{i\alpha} B_{\alpha j}\right) = \sum_{\alpha=1}^{c(A)} (\Phi_{\alpha ij} A_{i\alpha} B_{\alpha j}) = \sum_{\alpha=1}^{c(A)} (A_{i\alpha} \Phi_{\alpha ij} B_{\alpha j})$$

$$= \sum_{\alpha=1}^{c(A)} \mathbf{A}_\alpha \mathbf{\Phi}_\alpha \mathbf{B}_\alpha$$

Because this is a newly defined entity, it is altogether appropriate to present and work out examples of weighted matrix products in full detail.

Example#1:

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 2 \end{pmatrix} \quad , \quad \mathbf{B} \equiv \begin{pmatrix} 0 & -1 & -2 \\ -3 & -4 & -5 \end{pmatrix}$$

$$\mathbf{\Phi}_0 \equiv \begin{pmatrix} 6 & 7 & 8 \end{pmatrix} \quad , \quad \mathbf{\Phi}_1 \equiv \begin{pmatrix} 9 & 10 & 11 \end{pmatrix}$$

then:

$$\begin{aligned}
(\mathbf{A} \circ \mathbf{B})_{00} &= \Phi_{000} A_{00} B_{00} + \Phi_{100} A_{01} B_{10} \\
&= (6 \bullet 1 \bullet 0) + (9 \bullet 2 \bullet (-3)) = 0 - 54 = -54 \\
(\mathbf{A} \circ \mathbf{B})_{01} &= \Phi_{001} A_{00} B_{01} + \Phi_{101} A_{01} B_{11} \\
&= (7 \bullet 1 \bullet (-1)) + (10 \bullet 2 \bullet (-4)) = -7 - 80 = -87 \\
(\mathbf{A} \circ \mathbf{B})_{02} &= \Phi_{002} A_{00} B_{02} + \Phi_{102} A_{01} B_{12} \\
&= (8 \bullet 1 \bullet (-2)) + (11 \bullet 2 \bullet (-5)) = -16 - 110 = -126
\end{aligned}$$

$\therefore$

$$\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} -54 & -87 & -126 \end{pmatrix}$$

or:

$$\begin{aligned}
\mathbf{A} \circ \mathbf{B} &= (1) \begin{pmatrix} 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + (2) \begin{pmatrix} 9 & 10 & 11 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -5 \end{pmatrix} \\
&= (1) \begin{pmatrix} 0 & -7 & -16 \end{pmatrix} + (2) \begin{pmatrix} -27 & -40 & -55 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -7 & -16 \end{pmatrix} + \begin{pmatrix} -54 & -80 & -110 \end{pmatrix} = \begin{pmatrix} -54 & -87 & -126 \end{pmatrix} \quad \checkmark
\end{aligned}$$

Example#2:

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} \quad , \quad \mathbf{B} \equiv \begin{pmatrix} 0 & 2 & 4 & 8 \\ 1 & 3 & 5 & 7 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{\Phi}_0 \equiv \begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \end{pmatrix} \quad , \quad \mathbf{\Phi}_1 \equiv \begin{pmatrix} 8 & 10 & 12 & 14 \\ 9 & 11 & 13 & 15 \end{pmatrix} \quad , \quad \mathbf{\Phi}_2 \equiv \begin{pmatrix} 16 & 18 & 20 & 22 \\ 17 & 19 & 21 & 23 \end{pmatrix}$$

then:

$$\begin{aligned}
(\mathbf{A} \circ \mathbf{B})_{00} &= \Phi_{000} A_{00} B_{00} + \Phi_{100} A_{01} B_{10} + \Phi_{200} A_{02} B_{20} \\
&= (0 \bullet 1 \bullet 0) + (8 \bullet 2 \bullet 1) + (16 \bullet 3 \bullet 1) = 0 + 16 + 48 = 64 \\
(\mathbf{A} \circ \mathbf{B})_{01} &= \Phi_{001} A_{00} B_{01} + \Phi_{101} A_{01} B_{11} + \Phi_{201} A_{02} B_{21} \\
&= (2 \bullet 1 \bullet 2) + (10 \bullet 2 \bullet 3) + (18 \bullet 3 \bullet 0) = 4 + 60 + 0 = 64 \\
(\mathbf{A} \circ \mathbf{B})_{02} &= \Phi_{002} A_{00} B_{02} + \Phi_{102} A_{01} B_{12} + \Phi_{202} A_{02} B_{22} \\
&= (4 \bullet 1 \bullet 4) + (12 \bullet 2 \bullet 5) + (20 \bullet 3 \bullet -1) = 16 + 120 - 60 = 76 \\
(\mathbf{A} \circ \mathbf{B})_{03} &= \Phi_{003} A_{00} B_{03} + \Phi_{103} A_{01} B_{13} + \Phi_{203} A_{02} B_{23} \\
&= (6 \bullet 1 \bullet 8) + (14 \bullet 2 \bullet 7) + (22 \bullet 3 \bullet 0) = 48 + 196 + 0 = 244 \\
(\mathbf{A} \circ \mathbf{B})_{10} &= \Phi_{010} A_{10} B_{00} + \Phi_{110} A_{11} B_{10} + \Phi_{210} A_{12} B_{20} \\
&= (1 \bullet 5 \bullet 0) + (9 \bullet 6 \bullet 1) + (17 \bullet 7 \bullet 1) = 0 + 54 + 119 = 173 \\
(\mathbf{A} \circ \mathbf{B})_{11} &= \Phi_{011} A_{10} B_{01} + \Phi_{111} A_{11} B_{11} + \Phi_{211} A_{12} B_{21} \\
&= (3 \bullet 5 \bullet 2) + (11 \bullet 6 \bullet 3) + (19 \bullet 7 \bullet 0) = 30 + 198 + 0 = 228 \\
(\mathbf{A} \circ \mathbf{B})_{12} &= \Phi_{012} A_{10} B_{02} + \Phi_{112} A_{11} B_{12} + \Phi_{212} A_{12} B_{22} \\
&= (5 \bullet 5 \bullet 4) + (13 \bullet 6 \bullet 5) + (21 \bullet 7 \bullet -1) = 100 + 390 - 147 = 343
\end{aligned}$$

$$(\mathbf{A} \circ \mathbf{B})_{13} = \Phi_{013} A_{10} B_{03} + \Phi_{113} A_{11} B_{13} + \Phi_{213} A_{12} B_{23}$$
$$= (7 \bullet 5 \bullet 8) + (15 \bullet 6 \bullet 7) + (23 \bullet 7 \bullet 0) = 280 + 630 + 0 = 910$$

$\therefore$

$$\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 64 & 64 & 76 & 244 \\ 173 & 228 & 343 & 910 \end{pmatrix}$$

or:

$$\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} +$$
$$+ \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 8 & 10 & 12 & 14 \\ 9 & 11 & 13 & 15 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} +$$
$$+ \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 16 & 18 & 20 & 22 \\ 17 & 19 & 21 & 23 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 2 & 4 & 6 \\ 5 & 15 & 25 & 35 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} +$$
$$+ \begin{pmatrix} 16 & 20 & 24 & 28 \\ 54 & 66 & 78 & 90 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} +$$
$$+ \begin{pmatrix} 48 & 54 & 60 & 66 \\ 119 & 133 & 147 & 161 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 4 & 16 & 48 \\ 0 & 30 & 100 & 280 \end{pmatrix} + \begin{pmatrix} 16 & 60 & 120 & 196 \\ 54 & 198 & 390 & 630 \end{pmatrix}$$
$$+ \begin{pmatrix} 48 & 0 & -60 & 0 \\ 119 & 0 & -147 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 64 & 64 & 76 & 244 \\ 173 & 228 & 343 & 910 \end{pmatrix} \quad \checkmark$$

The Weighted Matrix product is especially useful in developing matrix bases closed under a (not necessarily associative) product.

As an example, consider the following developments:

Two Dimensional Dot and Cross Products

$$\mathbf{u}^{2;0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \mathbf{u}^{2;1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\mathbf{u}^{2;i} \circ \mathbf{u}^{2;j}$ is weighted matrix multiplication,

$$(a_{ij})(b_{ij}) = \left(\sum_h \Phi_{hij} a_{ih} b_{hj} \right)$$

with weights: $(\mathbf{\Phi}_h \equiv \boldsymbol{\varphi}^{2;h})$

$$\boldsymbol{\varphi}^{2;0} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \quad , \quad \boldsymbol{\varphi}^{2;1} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{u}^{2;0} \circ \mathbf{u}^{2;0} = \begin{pmatrix} [(-1)\bullet 1 \bullet 1 + (-1)\bullet 0 \bullet 0] & [1 \bullet 1 \bullet 0 + (-1)\bullet 0 \bullet 1] \\ [(-1)\bullet 0 \bullet 1 + 1 \bullet 1 \bullet 0] & [(-1)\bullet 0 \bullet 0 + (-1)\bullet 1 \bullet 1] \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{u}^{2;0} \circ \mathbf{u}^{2;1} = \begin{pmatrix} [(-1)\bullet 1 \bullet 0 + (-1)\bullet 0 \bullet 1] & [1 \bullet 1 \bullet 1 + (-1)\bullet 0 \bullet 0] \\ [(-1)\bullet 0 \bullet 0 + 1 \bullet 1 \bullet 1] & [(-1)\bullet 0 \bullet 1 + (-1)\bullet 1 \bullet 0] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$\mathbf{u}^{2;1} \circ \mathbf{u}^{2;0} = \begin{pmatrix} [(-1)\bullet 0 \bullet 1 + (-1)\bullet 1 \bullet 0] & [1 \bullet 0 \bullet 0 + (-1)\bullet 1 \bullet 1] \\ [(-1)\bullet 1 \bullet 1 + 1 \bullet 0 \bullet 0] & [(-1)\bullet 1 \bullet 0 + (-1)\bullet 0 \bullet 1] \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{u}^{2;1} \circ \mathbf{u}^{2;1} = \begin{pmatrix} [(-1)\bullet 0 \bullet 0 + (-1)\bullet 1 \bullet 1] & [1 \bullet 0 \bullet 1 + (-1)\bullet 1 \bullet 0] \\ [(-1)\bullet 1 \bullet 0 + 1 \bullet 0 \bullet 1] & [(-1)\bullet 1 \bullet 1 + (-1)\bullet 0 \bullet 0] \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\mathbf{u}^{2;0} \circ \mathbf{u}^{2;0} = -\mathbf{u}^{2;0}$	$\mathbf{u}^{2;1} \circ \mathbf{u}^{2;0} = -\mathbf{u}^{2;1}$
$\mathbf{u}^{2;0} \circ \mathbf{u}^{2;1} = \mathbf{u}^{2;1}$	$\mathbf{u}^{2;1} \circ \mathbf{u}^{2;1} = -\mathbf{u}^{2;0}$

so,with:

$$\mathbf{A} \equiv \mathbf{u}^{2;0} A^0 + \mathbf{u}^{2;1} A^1 \quad , \quad \mathbf{B} \equiv \mathbf{u}^{2;0} B^0 + \mathbf{u}^{2;1} B^1$$

then:

$$\mathbf{A} \circ \mathbf{B} = \mathbf{u}^{2;0}(-A^0 B^0 - A^1 B^1) + \mathbf{u}^{2;1}(A^0 B^1 - A^1 B^0)$$

and, so:

$$\mathbf{A} \odot \mathbf{B} \equiv \tfrac{1}{2}(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A}) = \mathbf{u}^{2;0}(-A^0 B^0 - A^1 B^1) \quad \text{, inner product}$$

$$\mathbf{A} \times \mathbf{B} \equiv \tfrac{1}{2}(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) = \mathbf{u}^{2;1}(A^0 B^1 - A^1 B^0) \quad \text{, outer product}$$

$$\mathbf{A} \circ (-\mathbf{u}^{2;0}) = \mathbf{A} \ , \ \forall \mathbf{A}$$

so:

$$\exists I_R \ (= -\mathbf{u}^{2;0}) \ni \mathbf{A} \circ I_R = \mathbf{A} \ , \ \forall \mathbf{A} \qquad \text{: Right Hand Identity}$$

$\therefore$

$\mathbf{A} \bullet \mathbf{B} \equiv D \in \mathbf{R} \ni \mathbf{A} \odot \mathbf{B} = I_R \bullet D$: dot product

$= A^0B^0 + A^1B^1$

[Note: the inner product, as defined here, has the usual properties of:
[Symmetry: $\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$
[Additivity: $(\mathbf{A} + \mathbf{B}) \odot \mathbf{C} = \mathbf{A} \odot \mathbf{C} + \mathbf{B} \odot \mathbf{C}$
[Homogeneity: $(k\mathbf{A}) \odot \mathbf{C} = k(\mathbf{A} \odot \mathbf{C})$
[The dot product, as defined here, adds the usual property:
[Positivity: $\mathbf{A} \bullet \mathbf{A} \geq 0 \ , \ \forall \mathbf{A}$ and $\mathbf{A} \bullet \mathbf{A} = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}$

and, so:

$\mathbf{A} \circ \mathbf{B} = \mathbf{A} \times \mathbf{B} + \mathbf{B} \odot \mathbf{A}$

$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

$\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$

$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$

Four Dimensional Dot and Cross Products

$$\mathbf{u}^{4;0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \mathbf{u}^{4;1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ,$$

$$\mathbf{u}^{4;2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad \mathbf{u}^{4;3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{u}^{4;i} \circ \mathbf{u}^{4;j}$ is weighted matrix multiplication,

$$(a_{ij})(b_{ij}) = \left(\sum_h \Phi_{hij} a_{ih} b_{hj} \right)$$

with weights: $(\mathbf{\Phi}_h \equiv \boldsymbol{\varphi}^{4;h})$

$$\boldsymbol{\varphi}^{4;0} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix} , \quad \boldsymbol{\varphi}^{4;1} = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix} ,$$

$$\boldsymbol{\varphi}^{4;2} = \begin{pmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix} , \quad \boldsymbol{\varphi}^{4;3} = \begin{pmatrix} -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

so, under weighted matrix multiplication[1] with

bases: $\mathbf{u}^{4;i}$, and weights: $\boldsymbol{\varphi}^{4;j}$:

$$\begin{array}{llll} \mathbf{u}^{4;0}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;1}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;1} , & \mathbf{u}^{4;2}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;2} , & \mathbf{u}^{4;3}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;3} \\ \mathbf{u}^{4;0}\circ\mathbf{u}^{4;1} = \mathbf{u}^{4;1} , & \mathbf{u}^{4;1}\circ\mathbf{u}^{4;1} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;2}\circ\mathbf{u}^{4;1} = -\mathbf{u}^{4;3} , & \mathbf{u}^{4;3}\circ\mathbf{u}^{4;1} = \mathbf{u}^{4;2} \\ \mathbf{u}^{4;0}\circ\mathbf{u}^{4;2} = \mathbf{u}^{4;2} , & \mathbf{u}^{4;1}\circ\mathbf{u}^{4;2} = \mathbf{u}^{4;3} , & \mathbf{u}^{4;2}\circ\mathbf{u}^{4;2} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;3}\circ\mathbf{u}^{4;2} = -\mathbf{u}^{4;1} \\ \mathbf{u}^{4;0}\circ\mathbf{u}^{4;3} = \mathbf{u}^{4;3} , & \mathbf{u}^{4;1}\circ\mathbf{u}^{4;3} = -\mathbf{u}^{4;2} , & \mathbf{u}^{4;2}\circ\mathbf{u}^{4;3} = \mathbf{u}^{4;1} , & \mathbf{u}^{4;3}\circ\mathbf{u}^{4;3} = -\mathbf{u}^{4;0} \end{array}$$

$\mathbf{u}^{4;0}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;0}\circ\mathbf{u}^{4;1} = \mathbf{u}^{4;1}$ $\mathbf{u}^{4;0}\circ\mathbf{u}^{4;2} = \mathbf{u}^{4;2}$ $\mathbf{u}^{4;0}\circ\mathbf{u}^{4;3} = \mathbf{u}^{4;3}$	$\mathbf{u}^{4;1}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;1}\circ\mathbf{u}^{4;1} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;1}\circ\mathbf{u}^{4;2} = \mathbf{u}^{4;3}$ $\mathbf{u}^{4;1}\circ\mathbf{u}^{4;3} = -\mathbf{u}^{4;2}$
$\mathbf{u}^{4;2}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;2}$ $\mathbf{u}^{4;2}\circ\mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;2}\circ\mathbf{u}^{4;2} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;2}\circ\mathbf{u}^{4;3} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;3}\circ\mathbf{u}^{4;0} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;3}\circ\mathbf{u}^{4;1} = \mathbf{u}^{4;2}$ $\mathbf{u}^{4;3}\circ\mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;3}\circ\mathbf{u}^{4;3} = -\mathbf{u}^{4;0}$

so,with:

$$\mathbf{A} \equiv \mathbf{u}^{4;0}A^0 + \mathbf{u}^{4;1}A^1 + \mathbf{u}^{4;2}A^2 + \mathbf{u}^{4;3}A^3$$
$$\mathbf{B} \equiv \mathbf{u}^{4;0}B^0 + \mathbf{u}^{4;1}B^1 + \mathbf{u}^{4;2}B^2 + \mathbf{u}^{4;3}B^3$$

then:

$$\begin{aligned} \mathbf{A}\circ\mathbf{B} = {} & \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) + \\ & + \mathbf{u}^{4;0}(-A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3) \end{aligned}$$

and, so:

$$\mathbf{A}\odot\mathbf{B} \equiv ½(\mathbf{A}\circ\mathbf{B} + \mathbf{B}\circ\mathbf{A}) = \mathbf{u}^{4;0}(-A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3) \quad ,\textit{inner product}$$
$$\begin{aligned} \mathbf{A}\times\mathbf{B} \equiv ½(\mathbf{A}\circ\mathbf{B} - \mathbf{B}\circ\mathbf{A}) = {} & \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) \quad ,\textit{outer product} \end{aligned}$$

$$\mathbf{A}\circ(-\mathbf{u}^{4;0}) = \mathbf{A} \ , \ \forall\mathbf{A}$$

so:

$$\exists I_R\ (= -\mathbf{u}^{4;0}) \ni \mathbf{A}\circ I_R = \mathbf{A} \ , \ \forall\mathbf{A} \qquad \text{: Right Hand Identity}$$

$\therefore$

$$\begin{aligned} \mathbf{A}\bullet\mathbf{B} \equiv {} & D \in \mathbf{R} \ni \mathbf{A}\odot\mathbf{B} = I_R \bullet D \qquad \text{: dot product} \\ = {} & A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \end{aligned}$$

[Note: the inner product, as defined here, has the usual properties of:
[Symmetry: $\mathbf{A}\odot\mathbf{B} = \mathbf{B}\odot\mathbf{A}$
[Additivity: $(\mathbf{A}+\mathbf{B})\odot\mathbf{C} = \mathbf{A}\odot\mathbf{C} + \mathbf{B}\odot\mathbf{C}$
[Homogeneity: $(k\mathbf{A})\odot\mathbf{C} = k(\mathbf{A}\odot\mathbf{C})$
[The dot product, as defined here, adds the usual property:
[Positivity: $\mathbf{A}\bullet\mathbf{A} \geq 0 \ , \ \forall\mathbf{A}$ and $\mathbf{A}\bullet\mathbf{A} = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}$

and, so:

$$\mathbf{A}\circ\mathbf{B} = \mathbf{A}\times\mathbf{B} + \mathbf{B}\odot\mathbf{A}$$
$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$
$$\mathbf{A}\odot\mathbf{B} = \mathbf{B}\odot\mathbf{A}$$

$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$

Note:

the inner and outer product definitions:

$\mathbf{A} \odot \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A})$

$\mathbf{A} \times \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A})$

are basis independant, from which arise the dot and cross product, which reduce to their ordinary forms in 3-D.

dot product:

$\mathbf{A} \bullet \mathbf{B} = A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$

Note, for 3-D ($A^0 = B^0 = 0$):

$$\mathbf{A} \bullet \mathbf{B} \equiv D \in R \ni DI_R = ½(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A}) = A^1B^1 + A^2B^2 + A^3B^3$$

the ordinary 3-D form

cross product:

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &\equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) \\ &= \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ &+ \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ &+ \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0)\end{aligned}$$

Note, for 3-D ($A^0 = B^0 = 0$):

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &\equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) \\ &= \mathbf{u}^{4;1}(A^2B^3 - A^3B^2) + \\ &+ \mathbf{u}^{4;2}(-A^1B^3 + A^3B^1) + \\ &+ \mathbf{u}^{4;3}(A^1B^2 - A^2B^1)\end{aligned}$$

the ordinary 3-D form

Clearly, $\mathbf{u}^{4;1} \equiv \hat{\mathbf{i}}$, $\mathbf{u}^{4;2} \equiv \hat{\mathbf{j}}$, $\mathbf{u}^{4;3} \equiv \hat{\mathbf{k}}$

it is a simple matter to show the linear independance, 3-D spanning, and the relations:

$$\begin{aligned}&\hat{\mathbf{i}} \bullet \hat{\mathbf{i}} = 1 ,\ \hat{\mathbf{i}} \bullet \hat{\mathbf{j}} = 0 ,\ \hat{\mathbf{i}} \bullet \hat{\mathbf{k}} = 0 , \\ &\hat{\mathbf{j}} \bullet \hat{\mathbf{i}} = 0 ,\ \hat{\mathbf{j}} \bullet \hat{\mathbf{j}} = 1 ,\ \hat{\mathbf{j}} \bullet \hat{\mathbf{k}} = 0 , \\ &\hat{\mathbf{k}} \bullet \hat{\mathbf{i}} = 0 ,\ \hat{\mathbf{k}} \bullet \hat{\mathbf{j}} = 0 ,\ \hat{\mathbf{k}} \bullet \hat{\mathbf{k}} = 1 , \\ &\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0} ,\ \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} ,\ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} , \\ &\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} ,\ \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0} ,\ \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} , \\ &\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} ,\ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} ,\ \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}\end{aligned}$$

A variation on the above is, with the same basis matrices, but with weights: ($\Phi_h \equiv \varphi^{4;h}$)

$$\varphi_1^{4;0} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix} \quad , \quad \varphi_1^{4;1} = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix} \quad ,$$

$$\varphi_1^{4;2} = \begin{pmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix} \quad , \quad \varphi_1^{4;3} = \begin{pmatrix} -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

so, under weighted matrix multiplication[1] with
bases: $\mathbf{u}^{4;i}$, and weights: $\varphi^{4;j}$:

$$\begin{array}{llll}
\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;0} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;1} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;2} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;3} \\
\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;2} \\
\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1} \\
\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;2} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;0}
\end{array}$$

$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;0}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;2}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;2}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;2}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;0}$

so,with:

$$\mathbf{A} \equiv \mathbf{u}^{4;0}A^0 + \mathbf{u}^{4;1}A^1 + \mathbf{u}^{4;2}A^2 + \mathbf{u}^{4;3}A^3$$
$$\mathbf{B} \equiv \mathbf{u}^{4;0}B^0 + \mathbf{u}^{4;1}B^1 + \mathbf{u}^{4;2}B^2 + \mathbf{u}^{4;3}B^3$$

then:

$$\begin{aligned} \mathbf{A} \circ \mathbf{B} = {} & \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) + \\ & + \mathbf{u}^{4;0}(A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3) \end{aligned}$$

and, so:

$$\mathbf{A} \odot \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A}) = \mathbf{u}^{4;0}(A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3) \quad ,\textit{inner product}$$
$$\begin{aligned} \mathbf{A} \times \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) = {} & \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) \quad ,\textit{outer product} \end{aligned}$$

$$(\mathbf{u}^{4;0}) \circ \mathbf{A} = \mathbf{A} \ , \ \forall \mathbf{A}$$

so:

$$\exists I_L \ (= \mathbf{u}^{4;0}) \ni I_L \circ \mathbf{A} = \mathbf{A} \ , \ \forall \mathbf{A} \qquad \text{: Left Hand Identity}$$

∴

$$\begin{aligned} \mathbf{A} \bullet \mathbf{B} \equiv {} & D \in \mathbf{R} \ni \mathbf{A} \odot \mathbf{B} = I_L \bullet D \qquad \text{: dot product} \\ = {} & A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3 \end{aligned}$$

[Note: the inner product, as defined here, has the usual properties of:
[Symmetry: $\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$
[Additivity: $(\mathbf{A} + \mathbf{B}) \odot \mathbf{C} = \mathbf{A} \odot \mathbf{C} + \mathbf{B} \odot \mathbf{C}$

[Homogeneity: $(k\mathbf{A}) \odot \mathbf{C} = k(\mathbf{A} \odot \mathbf{C})$
[The dot product, as defined here, is not positive definite, but
[suggests a Lorentz-Minkowski pseudo-metric:
[$\mathbf{A} \bullet \mathbf{A}$ $(A^0 \equiv cA^t)$ positive definite in some sub-domain.

and, so:

$$\mathbf{A} \circ \mathbf{B} = \mathbf{A} \times \mathbf{B} + \mathbf{B} \odot \mathbf{A}$$
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$
$$\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$$
$$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$$

Note:

the inner and outer product definitions:

$$\mathbf{A} \odot \mathbf{B} \equiv \tfrac{1}{2}(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A})$$
$$\mathbf{A} \times \mathbf{B} \equiv \tfrac{1}{2}(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A})$$

are basis independant, from which arise the dot and cross product, which reduce to their ordinary forms in 3-D.

dot product:

$$\mathbf{A} \bullet \mathbf{B} = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3$$

Note, for 3-D $(A^0 = B^0 = 0)$:

$$\mathbf{A} \bullet \mathbf{B} \equiv D \in R \ni DI_R = \tfrac{1}{2}(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A})$$
$$= -A^1B^1 - A^2B^2 - A^3B^3$$

the ordinary 3-D form

cross product:

$$\mathbf{A} \times \mathbf{B} \equiv \tfrac{1}{2}(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A})$$
$$= \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) +$$
$$+ \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) +$$
$$+ \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0)$$

(unchanged)

Another variation on the above is, with the same basis matrices, but with weights: $(\Phi_h \equiv \varphi^{4;h})$

$$\varphi_2^{4;0} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} , \quad \varphi_2^{4;1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} ,$$

$$\varphi_2^{4;2} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} , \quad \varphi_2^{4;3} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

so, under weighted matrix multiplication[1] with bases: $\mathbf{u}^{4;i}$, and weights: $\varphi^{4;j}$:

$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;0}$, $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;1}$, $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;2}$, $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;3}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$, $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;0}$, $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$, $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;2}$

$$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2} \quad , \quad \mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3} \quad , \quad \mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;0} \quad , \quad \mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$$
$$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3} \quad , \quad \mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;2} \quad , \quad \mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1} \quad , \quad \mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;0}$$

$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;0}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;2}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;2}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;0}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;2}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;0}$

so,with:

$$\mathbf{A} \equiv \mathbf{u}^{4;0}A^0 + \mathbf{u}^{4;1}A^1 + \mathbf{u}^{4;2}A^2 + \mathbf{u}^{4;3}A^3$$
$$\mathbf{B} \equiv \mathbf{u}^{4;0}B^0 + \mathbf{u}^{4;1}B^1 + \mathbf{u}^{4;2}B^2 + \mathbf{u}^{4;3}B^3$$

then:

$$\begin{aligned}\mathbf{A} \circ \mathbf{B} = {} & \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) + \\ & + \mathbf{u}^{4;0}(-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3)\end{aligned}$$

and, so:

$$\mathbf{A} \odot \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A}) = \mathbf{u}^{4;0}(-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3) \quad \text{, *inner product*}$$
$$\begin{aligned}\mathbf{A} \times \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) = {} & \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) \quad \text{, outer product}\end{aligned}$$

$$\mathbf{A} \circ (-\mathbf{u}^{4;0}) = \mathbf{A} \quad , \quad \forall \mathbf{A}$$

so:

$$\exists I_R \, (= -\mathbf{u}^{4;0}) \ni \mathbf{A} \circ I_R = \mathbf{A} \quad , \quad \forall \mathbf{A}$$: Right Hand Identity

∴

$$\mathbf{A} \bullet \mathbf{B} \equiv D \in \mathbf{R} \ni \mathbf{A} \odot \mathbf{B} = I_R \bullet D$$: dot product
$$= A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3$$

[Note: the inner product, as defined here, has the usual properties of:
[Symmetry: $\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$
[Additivity: $(\mathbf{A} + \mathbf{B}) \odot \mathbf{C} = \mathbf{A} \odot \mathbf{C} + \mathbf{B} \odot \mathbf{C}$
[Homogeneity: $(k\mathbf{A}) \odot \mathbf{C} = k(\mathbf{A} \odot \mathbf{C})$
[The dot product, as defined here, is not positive definite, but
[suggests a Lorentz-Minkowski pseudo-metric:
[$\mathbf{A} \bullet \mathbf{A} \;\; (A^0 \equiv cA^t)$ positive definite in some sub-domain.

and, so:

$$\mathbf{A} \circ \mathbf{B} = \mathbf{A} \times \mathbf{B} + \mathbf{B} \odot \mathbf{A}$$
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$
$$\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$$

$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$

Note:

the inner and outer product definitions:

$\mathbf{A} \odot \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A})$

$\mathbf{A} \times \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A})$

are basis independant, from which arise the dot and cross product, which reduce to their ordinary forms in 3-D.

dot product:

$$\mathbf{A} \bullet \mathbf{B} = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3$$

Note, for 3-D ($A^0 = B^0 = 0$):

$$\mathbf{A} \bullet \mathbf{B} \equiv D \in R \ni DI_R = ½(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A})$$
$$= -A^1B^1 - A^2B^2 - A^3B^3$$

the ordinary 3-D form

cross product:

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &\equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) \\ &= \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ &+ \mathbf{u}^{4;2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1) + \\ &+ \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0)\end{aligned}$$

(unchanged)

With another variation on the above, we can create a "Left-Hand-Rule" space, where $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = \hat{\mathbf{j}}$, $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -\hat{\mathbf{j}}$, with the same basis matrices, but with weights: ($\Phi_h \equiv \varphi^{4;h}$)

$$\varphi_3^{4;0} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} , \quad \varphi_3^{4;1} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix} ,$$

$$\varphi_3^{4;2} = \begin{pmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} , \quad \varphi_3^{4;3} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

so, under weighted matrix multiplication[1] with

bases: $\mathbf{u}^{4;i}$, and weights: $\varphi^{4;j}$:

$$\begin{array}{llll} \mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;1} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;2} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;3} \\ \mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;2} \\ \mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;0} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1} \\ \mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3} , & \mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;2} , & \mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1} , & \mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;0} \end{array}$$

$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2}$ $\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3}$ $\mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;2}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;2}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;0}$ $\mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;0} = -\mathbf{u}^{4;3}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;2}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$ $\mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;0}$

so,with:

$$\mathbf{A} \equiv \mathbf{u}^{4;0}A^0 + \mathbf{u}^{4;1}A^1 + \mathbf{u}^{4;2}A^2 + \mathbf{u}^{4;3}A^3$$
$$\mathbf{B} \equiv \mathbf{u}^{4;0}B^0 + \mathbf{u}^{4;1}B^1 + \mathbf{u}^{4;2}B^2 + \mathbf{u}^{4;3}B^3$$

then:

$$\begin{aligned}\mathbf{A} \circ \mathbf{B} = & \ \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 + A^1B^3 - A^2B^0 - A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) + \\ & + \mathbf{u}^{4;0}(-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3)\end{aligned}$$

and, so:

$$\mathbf{A} \odot \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A}) = \mathbf{u}^{4;0}(-A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3) \quad , \textit{inner product}$$
$$\begin{aligned}\mathbf{A} \times \mathbf{B} \equiv ½(\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}) = & \ \mathbf{u}^{4;1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2) + \\ & + \mathbf{u}^{4;2}(A^0B^2 + A^1B^3 - A^2B^0 - A^3B^1) + \\ & + \mathbf{u}^{4;3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0) \quad , \textit{outer product}\end{aligned}$$

$$\mathbf{A} \circ (-\mathbf{u}^{4;0}) = \mathbf{A} \ , \ \forall \mathbf{A}$$

so:

$$\exists I_R \ (= -\mathbf{u}^{4;0}) \ni \mathbf{A} \circ I_R = \mathbf{A} \ , \ \forall \mathbf{A}$$: Right Hand Identity

$\therefore$

$$\mathbf{A} \bullet \mathbf{B} \equiv D \in \mathbf{R} \ni \mathbf{A} \odot \mathbf{B} = I_R \bullet D$$: dot product
$$= A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3$$

[Note: the inner product, as defined here, has the usual properties of:
[Symmetry: $\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$
[Additivity: $(\mathbf{A} + \mathbf{B}) \odot \mathbf{C} = \mathbf{A} \odot \mathbf{C} + \mathbf{B} \odot \mathbf{C}$
[Homogeneity: $(k\mathbf{A}) \odot \mathbf{C} = k(\mathbf{A} \odot \mathbf{C})$
[The dot product, as defined here, adds the usual property:
[Positivity: $\mathbf{A} \bullet \mathbf{A} \geq 0 \ , \ \forall \mathbf{A}$ and $\mathbf{A} \bullet \mathbf{A} = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}$

and, so:

$$\mathbf{A} \circ \mathbf{B} = \mathbf{A} \times \mathbf{B} + \mathbf{B} \odot \mathbf{A}$$
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$
$$\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$$
$$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$$